CORRESPONDENCE

Comments on "The Depth-Dependent Current and Wave Interaction Equations: A Revision"

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ABSTRACT

Equations for the wave-averaged three-dimensional momentum equations have been published in this journal. It appears that these equations are not consistent with the known depth-integrated momentum balance, especially over a sloping bottom. These equations should thus be considered with caution, because they can produce erroneous flows, particularly outside of the surf zone. It is suggested that the inconsistency in the equations may arise from the different averaging operators applied to the different terms of the momentum equation. It is concluded that other forms of the momentum equations, expressed in terms of the quasi-Eulerian velocity, are better suited for three-dimensional modeling of wave–current interactions.

1. Introduction

The wave-averaged conservation of momentum can take essentially two forms, one for the total momentum, which includes the wave pseudomomentum (hereafter "wave momentum"; see McIntyre 1981), and the other for the mean flow momentum only. In terms of velocities, the first is associated with the Lagrangian velocity, whereas the second is associated with a quasi-Eulerian velocity introduced by Jenkins (1986). This is well known for depth-integrated equations (Longuet-Higgins and Stewart 1964; Garrett 1976; Smith 2006), but the vertical profiles of the mass and momentum balances are more complex. The pioneering effort of Mellor (2003, hereafter M03) produced practical wave-averaged equations for the total momentum that, in principle, may be used in primitive equation models to investigate coastal flows, such as the wave-driven circulations observed by Lentz et al. (2008). The first formulation (M03) was slightly inconsistent because of the improper approximation of wave motion with Airy wave theory. Indeed Airy theory is appropriate for most applications, but, for the expression of radiation stresses in three dimensions, it produces errors at the leading order, however small the slope may be. This question was discussed by Ardhuin et al. (2008b, hereafter ARB08), and a correction was given and verified. These authors acknowledged that the corrected equations, using the proper approximation, are not well suited for practical applications because very complex wave models are required for the correct estimation of the vertical fluxes of wave momentum, which are part of the fluxes of total momentum. Indeed, going beyond Airy theory requires solving Laplace's equation, which usually entails using phase-resolving models that couple various modes of motion (Athanassoulis and Belibassakis 1999; Chandrasekera and Cheung 2001). Such a model was used for wave propagation only over a 20 km^2 area around a submarine canyon with only five modes (Magne et al. 2007), which was already very costly in calculation time. Whereas the number of modes could be limited with the different choice of the vertical structure of the modes, ARB08 showed that the same model may need at least 10 modes to converge close enough to the solution.

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Although M03 gave correct wave-forcing expressions—in terms of velocity, pressure, and wave-induced displacement, before any approximation—Mellor (2008, hereafter M08) derived a new and different solution from scratch. The two theories may be consistent over a flat bottom, but they differ at their lowest order over sloping bottoms, so that the M08 equations are likely to be flawed, given the analysis of M03 by ARB08 and the fact that their consistency was not verified numerically over sloping bottoms.

Instead, M08 asserted that the equations are consistent with the depth-integrated equations of Phillips (1977). Further, about the test case proposed by ARB08, M08 stated that the wave energy was unchanged along the wave propagation and that the resulting wave forcing should be uniform over the depth. Here, we show that the M08 equations do not yield the known depth-integrated equations (Phillips 1977) with a difference that produces very different mean sea level variations when waves propagate over a sloping bottom. As for the test case proposed by ARB08, we show that a consistent analysis should take into account the small but significant change in wave energy due to shoaling. In the absence of dissipative processes, the M08 equations can produce spurious velocities of at least 30 cm s^{-1}, with 1-m-high waves over a bottom slope on the order of 1% in 4-m water depth.

2. Depth integration of the M08 equations

For simplicity we consider motions limited to a vertical plane (x, z) with constant water density and no Coriolis force, wind stress, or bottom friction. The wave-averaged momentum equation in M08 takes the form

$$\frac{\partial U}{\partial t} + \frac{\partial U^2}{\partial x} + \frac{\partial UW}{\partial z} = -g\frac{\partial \hat{\eta}}{\partial x} + F, \tag{1}$$

and the continuity equation is

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \tag{2}$$

where U and W are the Lagrangian mean velocity components, which contain the current and Stokes drift velocities; g is the acceleration due to gravity; and $\hat{\eta}$ is the time-averaged water level at the horizontal position x. The correctness of Eq. (2) crucially depends on the unity of the Jacobian of the averaging operator, which is the case for M03 [for a discussion of this with the generalized Lagrangian mean, see McIntyre (1981) and Ardhuin et al. (2008a)]. The force given by M08 on the right-hand side of (1) can be written as the sum

$$F = F_{px}^{\mathrm{M08}} + F_{uu} \tag{3}$$

of a wave-induced pressure gradient,

$$F_{px}^{\mathrm{M08}} = -\frac{\partial S_{px}^{\mathrm{M08}}}{\partial x} = -\frac{\partial}{\partial x}(E_D - \overline{\tilde{w}^2}) \tag{4}$$

$$\simeq -\frac{\partial}{\partial x}(E_D - kEF_{\mathrm{SC}}F_{\mathrm{SS}}), \tag{5}$$

and the divergences of the horizontal flux of wave momentum,

$$F_{uu} = -\frac{\partial S_{uu}}{\partial x} = -\frac{\partial \overline{\tilde{u}^2}}{\partial x} \tag{6}$$

$$\simeq -\frac{\partial}{\partial x}(kEF_{\mathrm{CC}}F_{\mathrm{CS}}), \tag{7}$$

where E is the wave energy; k is the wavenumber; and $\tilde{u}$ and $\tilde{w}$ are the horizontal and vertical wave-induced (orbital) velocities, respectively. Here, E_D is defined by a delta function, $E_D = \delta(z - \hat{\eta})E/2$: namely,

$$E_D = 0 \quad \text{if} \quad z \neq \hat{\eta} \quad \text{and} \quad \int_{-h}^{\hat{\eta}^+} E_D\,dz = \frac{E}{2}. \tag{8}$$

Using the bottom elevation $-h$ and mean water depth $D = \hat{\eta} + h$, the nondimensional functions F_{CC}, F_{SS}, and F_{SC} of kz and kD are

$$F_{\mathrm{CC}} = \frac{\cosh(kz + kh)}{\cosh(kD)}, \tag{9}$$

$$F_{\mathrm{SS}} = \frac{\sinh(kz + kh)}{\sinh(kD)}, \quad \text{and} \tag{10}$$

$$F_{\mathrm{SC}} = \frac{\sinh(kz + kh)}{\cosh(kD)}. \tag{11}$$

Using the notations of Smith (2006), the depth-averaged mass transport velocity is $M^T/D = 1/D\int_{-h}^{\hat{\eta}} U\,dz$. The total momentum M^T, as defined by Phillips (1977), contains, by definition, all the momentum in the water column, and the vertical coordinate transform by M03 is a nice way of distributing that momentum over the vertical to obtain U. This momentum can be expressed as the sum of a mean flow momentum M^m and a wave momentum M^w, as further discussed by Garrett (1976) and Smith (2006).

M08 correctly noted that

$$\int_{-h}^{\hat{\eta}} \underbrace{(S_{px}^{\mathrm{M08}} + S_{uu})}_{S_{xx}^{\mathrm{M08}}}\,dz = S_{xx}^{\mathrm{P77}}, \tag{12}$$

with S_{xx}^{P77} given by Phillips [1977, Eq. (3.6.27)]. However, for a depth-uniform U, the depth-integrated momentum equation in Phillips [1977, Eq. (3.6.11)] is

$$\frac{\partial M^T}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^T M^T}{D}\right) = -gD\frac{\partial \hat{\eta}}{\partial x} + \frac{\partial}{\partial x} S_{xx}^{\mathrm{P77}}. \quad (13)$$

This is also discussed at length by Smith (2006). The exact place of term $M^w M^w/D$ inside or outside of the radiation stresses is irrelevant for our discussion; following M03, we shall neglect it on the grounds that is a fourth-order quantity, although this is not always a good reason, as discussed by Smith (2006).

The forcing in the depth integration of (1) differs from the forcing in (13), because the gradient is inside of the integral. We note that $S_{px}^{\mathrm{M08}}(-h) = 0$, and, in order to apply Leibniz' rule, which is only valid for continuous functions, we first approximate the delta function by

$$E_D = \lim_{K\to\infty} \frac{K}{1 - \exp(-KD)} \frac{E}{2} \exp[K(z - \hat{\eta})] \quad (14)$$

and properly take the limit $K \to \infty$ after computing the integral. This function gives the expected integral $E/2$ when integrated from $z = -h$ to $z = \hat{\eta}$, where $D = h + \hat{\eta}$, and develops a singularity at $z = -\hat{\eta}$ as K goes to infinity. For large enough K, we can take the approximation $K/[1 - \exp(-KD)] \simeq K$.

We thus isolate the delta function,

$$\int_{-h}^{\hat{\eta}} \frac{\partial S_{xx}^{\mathrm{M08}}}{\partial x} dz = \int_{-h}^{\hat{\eta}} \frac{\partial}{\partial x}(S_{xx}^{\mathrm{M08}} - E_D)\, dz + \int_{-h}^{\hat{\eta}} \frac{\partial E_D}{\partial x} dz. \quad (15)$$

For the first term, we use Eq. (12) and Leibniz's rule; for the second term, we must be careful that E_D is a function of the x coordinate, not only via the wave energy E but also via the mean sea level $\hat{\eta}$ in eq. (14), hence,

$$\begin{aligned}\int_{-h}^{\hat{\eta}} \frac{\partial S_{xx}^{\mathrm{M08}}}{\partial x} dz = {} & \frac{\partial}{\partial x}\left(S_{xx}^{\mathrm{P77}} - \frac{E}{2}\right) - S_{uu}^{\mathrm{M08}}(z = -h)\frac{\partial h}{\partial x} \\ & - (S_{xx}^{\mathrm{M08}} - E_D)|_{z=\hat{\eta}}\frac{\partial \hat{\eta}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{E}{2}\right) \\ & - \int_{-h}^{\hat{\eta}} \frac{E}{2} K^2 \frac{\partial \hat{\eta}}{\partial x} \exp[K(z - \hat{\eta})]\, dz, \quad (16)\end{aligned}$$

where the first line comes from the first term in the right-hand side of Eq. (15) and the second line comes from the second term. Collecting terms this gives

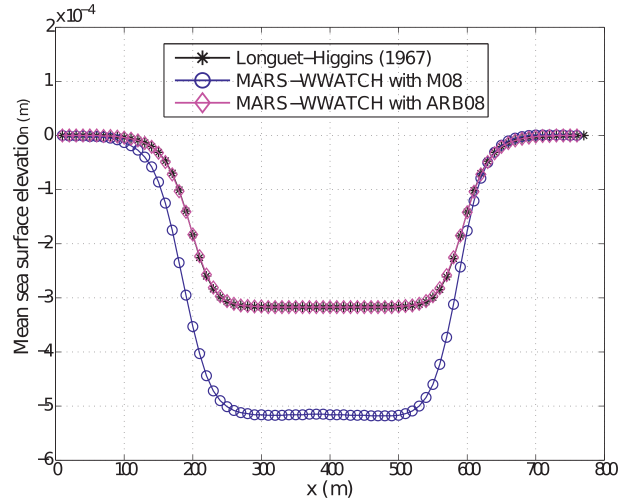

FIG. 1. Mean sea surface elevation induced by monochromatic waves propagating over the smooth bottom shown in Fig. 2, with amplitude H_s = 0.34 m and period T = 5.24 s. The extra forcing terms in Eq. (17) lead to an overestimation of the setdown by more than 50% for this case.

$$\begin{aligned}\int_{-h}^{\hat{\eta}} \frac{\partial S_{xx}^{\mathrm{M08}}}{\partial x} dz = {} & \frac{\partial S_{xx}^{\mathrm{P77}}}{\partial x} - S_{uu}^{\mathrm{M08}}(z = -h)\frac{\partial h}{\partial x} \\ & - \left\{(S_{xx}^{\mathrm{M08}} - E_D)|_{z=\hat{\eta}} \right. \\ & \left. + K\frac{E}{2}[1 - \exp(-KD)]\right\}\frac{\partial \hat{\eta}}{\partial x}. \quad (17)\end{aligned}$$

We thus have two extra terms compared to the Phillips (1977) expression.

Because of the last term, this integral clearly goes to infinity as K becomes very large, showing that the equations are not well defined. We could stop there, but this last term can be removed by redefining E_D as a delta function in sigma coordinates, which we shall do here. In that case, the only significant extra term is $S_{xx}^{\mathrm{M08}}(z = -h)\partial h/\partial x = -2kE(\partial h/\partial x)/\sinh(2kD)$, which can be dominant over a sloping bottom. As a result, the momentum balance in M08, unlike M03, does not produce the known setdown and setup. This is illustrated in Fig. 1. We take the case proposed by ARB08 with steady monochromatic waves shoaling on a slope without breaking or bottom friction and for an inviscid fluid, conditions in which exact numerical solutions are known. The bottom shoals smoothly from a depth of D = 6 m to D = 4 m. We added a symmetric slope back down to 6 m to allow periodic boundary conditions if needed. For a wave period of 5.24 s, the group velocity varies little from 4.89 to 4.64 m s^{-1}, giving a 2.7% increase of wave amplitude on the shoal. Contrary to statements in M08, $\partial E/\partial x$ is significant, with a 5.4% change of E over a few wavelengths.

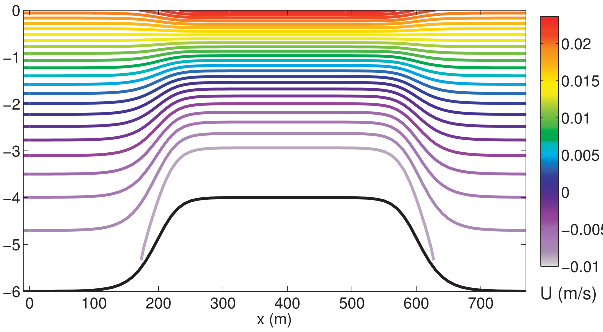

FIG. 2. Lagrangian velocity U for the inviscid sloping bottom case with $H_s = 1.02$ m and $T = 5.24$ s, obtained from the quasi-Eulerian analysis as $U = \hat{u} + U_s$. Contours are equally spaced from −0.01 to 0.025 m s^{-1}. The thick black line is the bottom elevation.

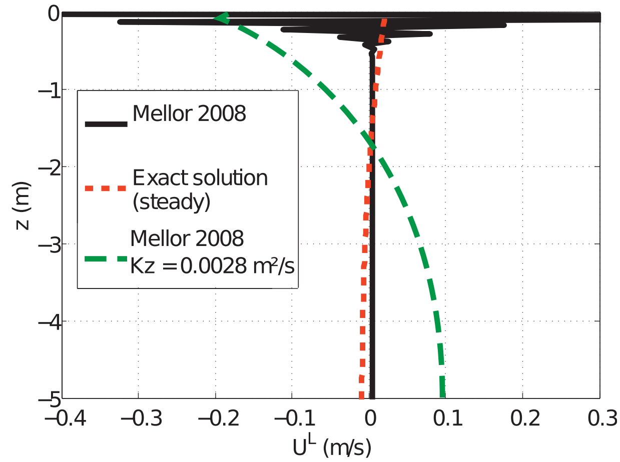

FIG. 3. Comparison of vertical profiles of U at $x = 200$ m given by different models: M08 without mixing (solid black line), M08 with mixing (dashed green line), and the exact solution (dashed red line). The wave parameters are $H_s = 1.02$ m and $T = 5.26$ s. All profiles are plotted after 6 min of time integration. The x axis was clipped, and the maximum velocities with M08 reached 0.8 m s^{-1}.

From the Eulerian analysis of that situation (e.g., Longuet-Higgins 1967), the mean water level should be 0.32 mm lower on the shoal (Fig. 1). Rivero and Arcilla (1995) established that there is no other dynamical effect: a steady Eulerian mean current develops, compensating for the divergence of the wave-induced mass transport (see also ARB08).

3. Flows produced by the M08 equations

In the correct solution, because the relative variation in phase speed is important, from 6.54 to 5.65 m s^{-1}, the Stokes drift accelerates on the shoal. The Eulerian velocity $\hat{u}$ is irrotational and thus nearly depth uniform, and it compensates the Stokes drift divergence by a convergence. The Lagrangian velocity U, shown in Fig. 2, is the sum of the two steady velocity fields.

We now solve for the equations derived by M08. The numerical solution is obtained by coupling the WAVEWATCH III wave model (Tolman 2009), solving the phase-averaged wave action equation, and the circulation Model for Applications at Regional Scales (MARS3D, Lazure and Dumas 2008). This coupling uses the generic coupler PALM (Buis et al. 2008). The feedback from flow to waves is negligible here and was thus turned off. MARS3D was implemented with 100 sigma levels regularly spaced and 5 active points in the transversal y direction, with 2 extra wall points and 2 ghost points needed to define finite differences; it is thus a real three-dimensional calculation, although the physical situation is two dimensional. There are 78 active points in the x direction. The time step was set to 0.05 s for tests with $H_s = 1.02$ m (1 s for $H_s = 0.34$ m). For simplicity, the wave model forcing is updated at each time step. We use Eq. (1) transformed to ς coordinates, with ς defined by $z = s(x, \varsigma, t) = \hat{\eta} + \varsigma D + \tilde{s}$ (M03),

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{W}{D}\frac{\partial U}{\partial \varsigma} = F - g\frac{\partial \hat{\eta}}{\partial x}, \qquad (18)$$

where the advection terms are obtained by using Eq. (2).

The flow boundary conditions are open. The monochromatic wave amplitude $a = 0.12$ m translates into a significant wave height H_s of 0.34 m for random waves with the same energy. We also test the model with $a =$ 0.36 m (i.e., $H_s = 1.02$ m), still far from the breaking limit in 4-m depth.

The discontinuity of the vertical profile in the forcing F, due to the E_D term, is not easily ingested by the numerical model and generates a strongly oscillating velocity profile (Fig. 3). These oscillations are absent at depths larger than 0.8 m, which is consistent with the zero values of F below the surface. A realistic constant viscosity $K_z = 2.8 \cdot 10^{-3}$ m^2 s^{-1} removes the oscillations and stabilizes the numerical calculations. However, this mixing only diffuses the negative term $-\int \partial E_D/\partial x\, dz = -0.5\partial E/\partial x$ over the vertical. That term is a source of momentum that produces velocities one order of magnitude larger than the Stokes drift U_s, with an opposite sign (Fig. 3). Thus, the problem is not just a question of delta function but one of a relatively large and spurious source of momentum.

The spurious velocities given by M08 with a realistic mixing are less for longer period waves: namely for $kD < 1$ (Table 1). They are comparable with those given by the M03 equations without mixing.

TABLE 1. Model results with M08: surface velocity at $x = 200$ m (on the upslope) for different model settings. The settings corresponding to the test in ARB08 are given in the second line. The surface velocity values are given for the time $t = 900$ s, except for the case without mixing ($t = 360$ s), which goes unstable earlier.

H_s (m)	T_p (s)	K_z (m^2 s^{-1})	U (m s^{-1})
1.02	5.6	0	0.6116
0.34	5.6	0	0.2127
0.34	13	0	0.3164
1.02	5.6	2.8×10^{-3}	−0.1594
0.34	5.6	2.8×10^{-3}	−0.0256
0.34	13	2.8×10^{-3}	−0.0007

4. Conclusions

We showed that the equations derived by M08 are inconsistent with the known depth-integrated momentum balances in the presence of a sloping bottom. In the absence of dissipation, a numerical integration of these equation produce unrealistic surface elevations and currents. The currents may reach significant values for very moderate waves, exceeding the expected results by one order of magnitude. Although we did not discuss the origin of the inconsistency, it appears that M08 used a different averaging for the pressure gradient term and for the advection terms of the same equation. We believe that this is the original reason for the problems discussed here. The spurious velocities produced by M08 are likely to be dwarfed by the strong forcing imposed by breaking waves in the surf zone. Nevertheless, we expect that the M08 equations can produce large errors for continental shelf applications, such as the investigation of cross-shore transports outside of the surf zone. Alternatively, equations for the quasi-Eulerian velocity as derived by McWilliams et al. (2004) and ARB08 can be used, which do not have such problems (Uchiyama et al. 2009). We thus encourage modelers of the coastal ocean to turn to this other form of the momentum equation.

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